APPENDIX 2 - EXPONENTIAL NOTATION AND LOGS

Exponential notation.
This is a convenient way of representing numbers which are very large or very small. For example, while it is convenient to write ten as 10 or one hundred as 100, larger numbers such as one million are often better written as $10^6$.
When a number is written in this form, it is called an exponential and it consists of a base (10) and an exponent (the power to which 10 is raised). Thus one million is represented as the base 10 raised to the power 6 when written as $10^6$. This is really $1 \times 10^6$, the 1 being called the pre-exponential term, but the 1 is often deleted.

Thus, in exponential notation, $1 \times 10^6$

To deduce the exponent, count the number of places that the decimal point must be moved to the left until the number becomes 1. Note that increasing the exponent by 1 actually increases the number by a factor of 10.

Examples: Starting with $1,000 = 10^3$,
\[ 10 \times 1000 = 10,000 = 10^4 \]
\[ 10 \times 10,000 = 100,000 = 10^5 \]
\[ 1,000,000,000,000 = 10^{12} \]

Note that any number raised to power 0 = 1, so $10^0 = 1$.
[See the next section for the proof of this.]

For numbers where the pre-exponential term is not 1, the pre-exponential term must be shown. Thus 120,000,000 can be written as $120 \times 10^6$ or $12 \times 10^7$ or as $1.2 \times 10^8$. In scientific notation, the pre-exponential term is usually written as less than 10 so that in this example it would be $1.2 \times 10^8$ or $1.20 \times 10^8$ or $1.200 \times 10^8$ etc, depending on how many significant figures one wishes to include.

Examples: $5600 = 5.6 \times 10^3$ (2 significant figures)
\[ 246000000 = 2.46 \times 10^8 \] (3 significant figures)

If more significant figures are required, they are included as zeros appended to the pre-exponential term. The exponent when it is an integer such as in the above examples is only a scale factor and gives no indication of the number of significant figures. Using the example above, $5.60 \times 10^3$ is 3 significant figures and $5.600 \times 10^3$ is 4 significant figures.
Similarly, small numbers such as one tenth are conveniently written as 0.1 for example. However a number such as one millionth (0.000,001) may be written as $10^{-6}$ or, more correctly, as $1 \times 10^{-6}$, with advantage. A number such as 0.000,000,12 can be written conveniently as $12 \times 10^{-8}$ or as $1.2 \times 10^{-7}$. The exponent required can be deduced by counting the number of places the decimal point must be moved to the right to obtain the pre–exponential component used and a minus sign must prefix the exponent.

Examples: \[ 0.0034 = 34 \times 10^{-4} \text{ (or } 3.4 \times 10^{-3}) \]
\[ 0.000,000,77 = 77 \times 10^{-8} \text{ (or } 7.7 \times 10^{-7}) \]

Illustrative Problem 1:
There are 602,200,000,000,000,000,000,000 atoms of carbon in 12 grams of carbon. Write this number using exponential notation.

Solution: No decimal point is shown and in this situation it is understood to be at the extreme right of the number (although such precision is most unlikely). Moving the decimal point to the left 23 places gives 6.022 as the pre-exponential term (to 4 significant figures) and $10^{23}$ as the exponential term. i.e. $6.022 \times 10^{23}$ atoms. Note that apart from the inconvenience of the length of writing the number 602200000000000000000000, the precision of this experimentally determined quantity is certainly not to 24 significant figures! Consequently, another advantage of using exponential notation is that it allows one to trim the pre-exponential term to an appropriate number of digits corresponding to the accuracy of the number. In this case, the best experimental data gives the number as $6.022136736 \times 10^{23}$.

Illustrative Problem 2:
Diamonds consist of pure carbon. If 12.0 grams of carbon contains $6.022 \times 10^{23}$ carbon atoms, how many carbon atoms are in a diamond which weighs 1.20 grams?

Solution:
Number of carbon atoms in 1.0 g of carbon = $\frac{6.022 \times 10^{23}}{12.0}$

\[ \therefore \text{ the number in } 1.20 \text{ g } = \frac{6.022 \times 10^{23}}{12.0} \times 1.20 = 0.602 \times 10^{23} \text{ atoms} \]

or $6.02 \times 10^{22}$ atoms.
Practice Questions.
Convert each of the following to the exponential form to base 10. Give 3 significant figures in the pre-exponential term in each case.

1. 441000000000 2. 3750000000000000000000
3. 0.00000000000000127 4. 0.000580
5. 100000000000000000000000 6. 0.00264

Answers. 1. 4.41 x $10^{11}$ 2. 3.75 x $10^{21}$ 3. 1.27 x $10^{-15}$
4. 5.80 x $10^{-4}$ 5. 1.00 x $10^{23}$ 6. 2.64 x $10^{-3}$

Multiplication and Division Using Exponentials.

(a) Multiplication
Example: $10^{6} \times 10^{3}$
Rule: Provided the exponentials have the same base, **add** the exponents of the numbers to be multiplied.

Solution: $10^{6} \times 10^{3} = 10^{(6+3)} = 10^{9}$

(b) Division
Example: $10^{6} \div 10^{2}$
Rule: Provided the exponentials have the same base, **subtract** the exponent of the denominator from the exponent of the numerator of the numbers being divided.

Solution: $10^{6} \div 10^{2} = 10^{(6-2)} = 10^{4}$

Be careful when the exponent on the bottom is negative:

e.g. $10^{8} \div 10^{-2} = 10^{(8-(-2))} = 10^{(8+2)} = 10^{10}$

**Proof that n^0 = 1:**
As an example, consider $2^{0}$.

This can be rewritten as $2^{0} = 2^{(1-1)} = \frac{2^{1}}{2^{1}}$ which cancels to = 1

The same is valid for any number, n.
Illustrative Problem 1:
The world's population was 6 billion ($6 \times 10^9$) people as of 2000. If each person on average consumes 1000 Calories per day, how many Calories are required to feed the entire population for 100 days?

Solution:
Total calories = number of people $\times$ calories per day $\times$ number of days
$= 6 \times 10^9 \times 1.000 \times 1.00 \times 10^2$
$= 6 \times 10^{(9+3+2)}$
$= 6 \times 10^{14}$ calories

Illustrative Problem 2:
If the total area of rainforest remaining on the planet $= 1 \times 10^6$ square kilometres and the population is $6 \times 10^9$ people, how many people are there on average per square kilometre of remaining rainforest?

Solution:
Persons per square kilometre $= \frac{\text{number of people}}{\text{number of square kilometres}}$
$= 6 \times 10^9 \div 1 \times 10^6$
$= 6 \times 10^{(9-6)}$
$= 6 \times 10^3$ people/square kilometre

If the pre-exponential component of more than one of the numbers is not 1, then the pre-exponential terms are multiplied or divided as a separate calculation. This is illustrated in the next example.

Illustrative Problem 3:
If each person creates on average $5 \times 10^3$ kg of garbage during a lifetime, what is the total garbage created by the current population of $6 \times 10^9$ people?

Solution:
Total garbage = average per person $\times$ number of people
$= 5 \times 10^3 \times 6 \times 10^9$
$= 6 \times 5 \times 10^{(3+9)}$
$= 30 \times 10^{12}$ or $3 \times 10^{13}$ kg/person.
Illustrative Problem 4:
If the total GDP of the world is $12 \times 10^{12}$, what is the average GDP per person?

Solution:
Average GDP = total GDP ÷ number of people

\[
= \frac{12 \times 10^{12}}{6 \times 10^{9}}
\]

\[
= \frac{12 \times 10^{12-9}}{6}
\]

\[
= 2 \times 10^{3} \text{ per person}
\]

Illustrative Problem 5:
Convert $2 \times 10^{-4}$ metre into micrometres (µm) given 1 micrometre = $10^{-6}$ metre.

Solution:
Number of micrometres = \[
\frac{2 \times 10^{-4}}{10^{-6}} = 2 \times 10^{2} \text{ micrometres}
\]

Practice Questions.

1. $10^{4} \times 10^{5} = \quad$ 2. $10^{15} ÷ 10^{4} = \quad$

3. $10^{7} \times 10^{-3} = \quad$ 4. $6.0 \times 10^{5} ÷ 2 \times 10^{2} = \quad$

5. $8 \times 10^{6} ÷ 4 \times 10^{-3} = \quad$ 6. $4 \times 10^{5} ÷ 2 \times 10^{-6} = \quad$

7. $3 \times 10^{-10} ÷ 1.5 \times 10^{-4} = \quad$ 8. $6 \times 10^{3} \times 2 \times 10^{-7} ÷ 4 \times 10^{-2} = \quad$

Answers.

1. $10^{9} \quad$ 2. $10^{11} \quad$ 3. $10^{10} \quad$ 4. $3 \times 10^{3}$

5. $2 \times 10^{11} \quad$ 6. $2 \times 10^{11} \quad$ 7. $2 \times 10^{-6} \quad$ 8. $3 \times 10^{-2}$

Logarithms.

Any number can be expressed in the exponential form without a pre-exponential term (or more precisely, with 1 for the pre-exponential term). The exponent is the logarithm (log for short) to the base ten such that 10 raised to that power equals the number. Thus \( \log_{10} n \) is the power to which 10 must be raised to equal \( n \) or \( 10^{\log_{10} n} = n \). The log of any number can most easily be obtained from basic calculators by entering the number and pressing the LOG key. Some calculators such as the Casio fx100s require the LOG key to be pressed first, enter the number and then press the = key.
e.g.  \( \log_{10} 10 = 1 \)  
\( \log_{10} 100 = 2 \)  
\( \log_{10} 1000 = 3 \)

i.e.  \( 10^1 = 10 \)  
\( 10^2 = 100 \)  
\( 10^3 = 1000 \)

However, in general the exponent is unlikely to be an integer and in this situation, additional rules apply. For example, using the log function of a calculator,
\( \log_{10} 2.0 = 0.30 \)  
i.e.  \( 10^{0.30} = 2.0 \)

**The log of any number less than 10 must have a log less than 1 as \( 10^1 = 10 \).**

A non-integer exponent or log is not a normal number but is made up of two components – the digits ahead of (to the left of) the decimal point (called the *characteristic*) and the digits following (to the right of) the decimal point (called the *mantissa*).

The digits to the left of the decimal point in the exponent that constitute the characteristic of the log are scale factors only – thus \( 10^2 = 100 \), \( 10^3 = 1000 \), etc. The digits to the right of the decimal in the exponent that constitute the mantissa of the log allow calculation of the number to which the scale factor must be applied. How many digits are to the right of the decimal (the mantissa) also gives the number of significant figures in the final number.

For example,  \( 10^{2.20} \) means \( 10^2 \times 10^{0.20} \)

The \( 10^2 \) term is easily evaluated mentally and = 100.

Evaluating the \( 10^{0.20} \) is done using the \( 10^x \) key of the calculator which gives 1.6 [see later on how to do this.]

Combining these terms, \( 10^2 \times 10^{0.20} = 100 \times 1.6 = 160 \).

Note that \( 10^{3.20} = 10^3 \times 10^{0.20} = 1000 \times 1.6 = 1600 \).

Similarly, \( 10^{4.20} = 16000 \),

and \( 10^{5.20} = 160000 \), etc.

However, note that in all these examples, the mantissa contained only 2 digits, 0.20, so only 2 significant figures should be quoted in the final answers which therefore should be trimmed to \( 1.6 \times 10^2 \), \( 1.6 \times 10^3 \), \( 1.6 \times 10^4 \) and \( 1.6 \times 10^5 \) respectively.

That the characteristic does not contribute to the number other than as a scale factor is seen in the following example:

The number 5600 could be written as \( 5.600 \times 10^3 \) or as \( 56.00 \times 10^2 \) or even as \( 560.0 \times 10^1 \). Using the log function of the calculator, 5600 is \( 10^{3.7482} \). If the number were written as \( 560.0 \times 10^1 \), by applying the log function to 560.0, this becomes \( 10^{2.7482} \times 10^1 \) or \( 10^{3.7482} \) when converted to the exponential form – the same as for 5600.
Similarly, $56.00 \times 10^2$ is $10^{1.7482} \times 10^2 = 10^{3.7482}$ and $5.600 \times 10^3$ is $10^{0.7482} \times 10^3 = 10^{3.7482}$ in their exponential forms – all are identical.

So, regardless of which scale factor is used, the mantissa indicates the digits 5600 and the characteristic gives the scale factor to use in conjunction with those digits.

To deduce the number of significant figures when a number is written in the exponential form, count the number of digits including all zeros following the decimal place in the mantissa.

Thus $10^{2.37}$ is two significant figures, $10^{4.1}$ is one significant figure and $10^{6.07}$ is two significant figures.

Practice Questions.
Obtain the logs of the following numbers. Give each to 3 decimal places. Thence express each number in the form $10^x$. What do you notice about the answers to questions 3, 4 and 5?

1. $2.483$  
2. $10.870$  
3. $194$  
4. $1940$  
5. $19400$

Answers.  
1. $0.395$; $10^{0.395}$  
2. $1.036$; $10^{1.036}$  
3. $2.288$; $10^{2.288}$  
4. $3.288$; $10^{3.288}$  
5. $4.288$; $10^{4.288}$

The answers to Q3 - Q5 increase by 1 sequentially because the numbers they represent have increased by a factor of 10 in sequence.

pH calculations.
Because the amount of $H^+$ in dilute water solutions of acids and bases is typically very small - e.g. in pure water, the concentration of $H^+ = 0.0000001$ moles per litre or $1 \times 10^{-7}$ M - it is convenient to express this as a log. The symbol $p$ is used as shorthand for “$-\log_{10}$” and the pH of a solution is $-\log_{10}[H^+]$. In other words, the pH is just the exponent when the concentration of hydrogen ions is put into the exponential form with the minus sign changed to make it a positive number. There are no units for pH as it is merely a number. As various approximations are made in the theory of pH, no more than 2 significant figures are quoted - i.e. 2 decimal places. Because pH is an exponential, only the digits following the decimal place are significant figures.

Practice Questions.
Express the following hydrogen ion concentrations as pH:

1. 0.015 M  
2. 0.000030 M  
3. 0.25 M  
4. 0.0025 M
Answers:
1. pH = \(-\log(0.015) = 1.82\)  
2. pH = \(-\log(0.000030) = 4.52\)  
3. pH = \(-\log(0.25) = 0.60\)  
4. pH = \(-\log(0.0025) = 2.60\)

Note how in questions 3 and 4 the concentrations changed by a factor of 100 but the pH only changed from 0.60 to 2.60. As explained above, this is because a change of a single pH unit corresponds to a tenfold change in hydrogen ion concentration. Failure to recognise this property of pH has lead to a common belief that because the pH of the ocean due to carbon dioxide pollution has only decreased by 0.3, this is not a matter of concern. In fact, this represents a 50% increase in hydrogen ion concentration with disastrous potential effects on marine life.

**Converting from the log form - use of the \(10^x\) function.**

To convert a logarithm back to a normal number, use the \(10^x\) key where \(x = \) the log to base 10. Usually the LOG key is the lower register of the \(10^x\) key because they perform opposite actions. Some calculators may have a \(\text{LOG}^{-1}\) key which performs the same operation as a \(10^x\) key. In either case, simply enter the value of \(x\) and press the \(10^x\) or \(\text{LOG}^{-1}\) key.

Some calculators such as the Casio fx100s require the following procedure: Enter 10, press SHIFT, enter the value of \(x\) followed by the +/- key if \(x\) is negative, press the = key.

**Practice Questions.**

Convert the following numbers from the exponential form. Ensure each answer has the correct number of significant figures.

1. \(10^{3.45}\)  
2. \(10^{3.450}\)  
3. \(10^{0.165}\)  
4. \(10^{-8.35}\)  
5. \(10^{-0.0200}\)

Answers.

Use the \(10^x\) button on the calculator.

1. \(2.8 \times 10^3\) (2 sig fig)  
2. \(2.82 \times 10^3\) (3 sig figs)  
3. 1.46 (3 sig figs)  
4. \(4.5 \times 10^{-9}\) (2 sig figs)  
5. \(9.55 \times 10^{-1}\) (3 sig figs)

**Converting pH to hydrogen ion concentration.**

To convert a pH into the hydrogen ion concentration, use the expression \([H^+] = 10^{-\text{pH}}\). The easiest way to do this as explained in the earlier section is to use the \(10^x\) button on the calculator.
Practice Questions.

Express each of the following solution pH values as a hydrogen ion concentration.

1. pH = 3.69  
2. pH = 5.62  
3. pH = 7.80  
4. pH = 1.58

Answers.
1. \([H^+] = 10^{-pH} = 10^{-3.69} \text{ M} = 0.00020 = 2.0 \times 10^{-4} \text{ M}.\]
2. \([H^+] = 10^{-pH} = 10^{-5.62} \text{ M} = 2.4 \times 10^{-6} \text{ M}.\]
3. \([H^+] = 10^{-pH} = 10^{-7.80} \text{ M} = 1.6 \times 10^{-8} \text{ M}.\]
4. \([H^+] = 10^{-pH} = 10^{-1.58} \text{ M} = 0.026 \text{ M} = 2.6 \times 10^{-2} \text{ M}.\]

Exponentials to base e.
The number represented by \( e \) (2.71828) is sometimes encountered in chemical calculations as the base for an exponential. Exactly the same rules apply to exponentials to base \( e \) as to base 10. There is an \( e^x \) key usually on the upper register of most calculators and it is used in the same way as the \( 10^x \) key. If \( e \) is the base used for a log, "ln" is used rather than "log e" and there is an ln key on your calculator.

Example: Give the value for \( e^{2.35} \).

Solution: Enter 2.35, press SHIFT and then the \( e^x \) key. Answer is 10.49

Example: Obtain ln(10.49)

Solution: Enter 10.49 and press the LN key. Answer is 2.35 (i.e. \( 10.49 = e^{2.35} \))

Practice Questions.
Use your calculator to evaluate the following:

1. \( e^{3.68} \)  
2. \( \ln 20.55 \)  
3. \( e^{-4.29} \)  
4. \( \ln 0.0225 \)  
5. \( e^{125.0} \)

Answers.
1. 39.65  
2. 3.023  
3. \( 1.37 \times 10^{-2} \)  
4. -3.794  
5. \( 1.94 \times 10^{54} \)